Statistical Analysis of Power Grid Networks Considering Lognormal Leakage Current Variations with Spatial Correlation

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***Abstract*— As the technology scales into 90nm and below, process-induced variations become more pronounced. In this paper, we propose an efficient stochastic method for analyzing the voltage drop variations of on-chip power grid networks, considering log-normal leakage current variations with spatial correlation. The new analysis is based on the Hermite polynomial chaos (PC) representation of random processes. Different from the existing Hermite PC based method for power grid analysis, which models all the random variations as Gaussian processes without considering spatial correlation. The new method focuses on the impacts of stochastic sub-threshold leakage currents, which are modeled as log-normal distribution random variables, on the power grid voltage variations. To consider the spatial correlation, we apply orthogonal decomposition to map the correlated random variables into independent variables. Our experiment results show that the new method is more accurate than the Gaussian-only Hermite PC method using the Taylor expansion method for analyzing leakage current variations, and two orders of magnitude faster than the Monte Carlo method with small variance errors. We also show that the spatial correlation may lead to large errors if not being considered in the statistical analysis.**

***Index Terms*— Power grid networks, Hermite polynomials, Principal component analysis, Spatial correlation**

I. Introduction

PROCESS-INDUCED variability has huge impacts on the circuit performance in the sub-90nm VLSI technologies [10], [9]. One important aspect of the variations comes from the chip leakage currents. Leakage currents come from different sources. The dominant factor is the sub-threshold leakage current. The reason is that sub-threshold leakage current has a rapid increasing rate (about 5X-10X increase per technology generation [2]), and it is highly sensitive to threshold voltage V_{th} variations, due to the exponential relationship between sub-threshold current I_{off} and threshold voltage V_{th} as shown below [14],

$$I_{off} = I_{s0} e^{\frac{V_{gs}-V_{th}}{nV_T}} (1 - e^{\frac{-V_{ds}}{V_T}}) \quad (1)$$

where I_{s0} is a constant related to the device characteristics, V_T is the thermal voltage, and n is a constant.

Clearly, the leakage current has exponential dependency on the threshold voltage V_{th}. In the sequel, the leakage current is mainly referred to as the sub-threshold leakage current. Detailed analysis shows that I_{off} is also an exponential function of the channel length L [12]. So, if we model V_{th} or L as the random variables with Gaussian variations due to inter-die or intra-die process variations, then the leakage currents will have a log-normal distribution as shown in [12]. On top of this, those random variables are spatially correlated within a die, due to the nature of the many physical and chemical manufacture processes [9].

Due to the importance of the impacts on leakage currents on the circuit performances, especially on the on-chip power delivery networks, a number of research works have been proposed recently to perform the stochastic analysis of power grid networks under process-induced leakage current variations. The voltage drop of power grid networks subject to the leakage current variations was first studied in [3], [4]. This method assumes that the log-normal distribution of the node voltage drop is due to log-normal leakage current inputs and is based on a localized Monte Carlo (sampling) method to compute the variance of the node voltage drop. However, this localized sampling method is limited to the static DC solution of power grids modeled as resistor-only networks. Therefore, it can only compute the responses to the standby leakage currents. However, the dynamic leakage currents become more significant, especially when the sleep transistors are intensively used nowadays for reducing leakage powers. In [13], [11], impulse responses are used to compute the means and variances of node voltage responses due to general current variations. But this method needs to know the impulse response from all the current sources to all the nodes, which is expensive to compute for a large network. In [12], the probability density function (pdf) of leakage currents are computed based on the Gaussian variations of channel length.

Recently, a stochastic simulation method for interconnect and power grid networks has been proposed [7], [15]. This method is based on the orthogonal polynomial chaos expansion of random processes to represent and solve for the stochastic responses of linear systems. The major benefit of this method is its compatibility with current transient simulation framework: it solves for some coefficients of the orthogonal polynomials, which can be done by using normal transient simulations of the original circuits with deterministic inputs to compute variances of node responses. Some existing

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approaches [7] model all the parameter variations as Gaussian (or approximate them as Gaussian variations by using first-order Taylor expansion) [15]. Those methods also fail to consider the spatial correlation in the process parameter random variables.

In this paper, we apply the orthogonal polynomial based methods (also called spectral statistical method) to deal with leakage current inputs with log-normal distributions and spatial correlations. We show how to represent a log-normal distribution in terms of Hermite polynomials, assuming Gaussian distribution of threshold voltage V_{th} in consideration of intra-die variation. To consider the spatial correlation, we apply orthogonal decomposition via principal component analysis to map the correlated random variables into independent variables. To the best knowledge of the authors, the proposed method is the first method being able to perform statistical analysis on power grids with variation dynamic leakage currents having log-normal distributions and spatial correlations. Experiment results show that the proposed method predicates the variances of the resulting log-normal-like node voltage drops more accurately than Taylor expansion based Gaussian approximation method.

II. Problem Formulation

A. Power Grid Network Models

The power grid networks in this paper are modeled as RC networks with known time-variant current sources, which are obtained by gate level logic simulations of the VLSI systems. For a power grid (versus the ground grid), some nodes have known voltage modeled as constant voltage sources. For C4 power grids, the known voltage nodes can be internal nodes inside the power grid. Given known deterministic vector of current sources, $I(t)$, the node voltages can be obtained by solving the following differential equations, which is formulated using modified nodal analysis (MNA) approach,

$$Gv(t) + C\frac{dv(t)}{dt} = I(t) \tag{2}$$

where G is the conductance matrix, C is admittance matrix resulting from capacitive elements. $v(t)$ is the vector of time-varying node voltages and branch currents of voltage sources that we try to solve.

B. Modeling Leakage Current Variations

The G and C matrices and input currents $I(t)$ depend on the circuit parameters, such as metal wire width, length, thickness on power grids, and transistor parameters, such as channel length, width, gate oxide thickness, etc. Some previous work assumes that all circuit parameters and current sources are treated as uncorrelated Gaussian random variables [7]. In this paper, we only consider the log-normal leakage current variation, due to the channel length variations, which is modeled as Gaussian (normal) variations [12]. All other circuit parameter variations can be easily considered as shown in [7].

Process-induced variations can also be classified into inter-die (die-to-die) variations and intra-die variations. In inter-die variations, all the parameters variations are correlated. The worst case corner can be easily found by setting the parameters to their top range (mean plus three standard deviations). The difficulty lies in the intra-die variations, where the circuit parameters are not correlated or spatially correlated. Intra-die variations also consist of local and layout dependent deterministic components and random components, which typically are modeled as multivariate Gaussian processes with any spatial correlation [1]. In this paper, we first assume we have a number of transformed ortho-normal random Gaussian variables $\xi(\theta), i = 1, ..., n$, which actually model the channel length and the device threshold voltage variations. After that, we consider spatial correlation in the intra-die variation. We apply the principal component analysis method to transfer the correlated variables into un-correlated variables before the spectral statistical analysis.

Let Θ denotes the process sampling space. Let $\theta \in \Theta$, $\xi_i : \theta \rightarrow R$ denotes a normalized Gaussian variable and $\xi(\theta) = [\xi_1(\theta), ..., \xi_n(\theta)]$ is a vector of n independent Gaussian variable. Therefore, given the process variations, the MNA for (2) becomes

$$Gv(t) + C\frac{dv(t)}{dt} = I(t, \xi(\theta)) \tag{3}$$

Note that the input current vector, $I(t, \xi(\theta))$, has both deterministic and random components. In this paper, we assume the dynamic currents (power) due to circuit switching are still modeled as deterministic currents as we only consider the leakage variations.

The problem we need solve is to efficiently find the mean and variances of voltage $v(t)$ at any node and at any time instance. A straightforward method is Monte Carlo (MC) based sampling methods. We randomly generate $I(t, \xi(\theta))$, which is based on the log-normal distribution, solve (3) in time domain for each sampling and compute the means and variances based on sufficient samplings. Obviously, MC will be computationally expensive. However, MC will give the most reliable results and is the most robust and flexible method.

III. Spectral Statistical based Simulation

A. Concept of Hermite Polynomial Chaos

In the following, a random variable $\xi(\theta)$ is expressed as a function of θ, which is the random event. Hermite PC utilizes a series of orthogonal polynomials (with respect to the Gaussian distribution) to facilitate stochastic analysis [16]. These polynomials are used as the orthogonal basis to decompose a random process in a similar way that sine and cosine functions are used to decompose a periodic signal in Fourier series expansion.

For a random variable $v(t, \xi)$ with limited variance, where $\xi = [\xi_1, \xi_2, ...\xi_n]$ is a vector of zero mean ortho-normal Gaussian random variables. The random variable can be approximated by truncated Hermite PC expansion as follows [6]:

$$v(t, \xi) = \sum_{k=0}^{P} a_k H_k^n(\xi) \tag{4}$$

where n is the number of independent random variables, $H_k^n(\xi)$ is n-dimensional Hermite polynomials and a_k are the deterministic coefficients. The number of terms P is given

$$P = \sum_{k=0}^{p} \frac{(n-1+k)!}{k!(n-1)!} \tag{5}$$

where p is the order of the Hermite PC. If only one random variable is considered, the one-dimensional Hermite polynomials are expressed as follows:

$$H_0^1(\xi) = 1, H_1^1(\xi) = \xi, H_2^1(\xi) = \xi^2 - 1, H_3^1(\xi) = \xi^3 - 3\xi, ... \tag{6}$$

Hermite polynomials are orthogonal with respect to Gaussian weighted expectation (the superscript n is dropped for simple notation):

$$< H_i(\xi), H_j(\xi) >=< H_i^2(\xi) > \delta_{ij} \tag{7}$$

where δ_{ij} is the Kronecker delta and $< *, * >$ denotes an inner product defined as follow:

$$< f(\xi), g(\xi) >= \frac{1}{\sqrt{(2\pi)^n}} \int f(\xi) g(\xi) e^{-\frac{1}{2}\xi^T \xi} d\xi \tag{8}$$

Like Fourier series, the coefficient a_k can be found by a projection operation onto the Hermite PC basis:

$$a_k(t) = \frac{< v(t,\xi), H_k(\xi) >}{< H_k^2(\xi) >}, \ \forall k \in \{0, ..., P\}. \tag{9}$$

B. Simulation Approach Based on Hermite PCs

In case that $v(t,\xi)$ is unknown random variable vector (with unknown distributions) like node voltages in (3), then the coefficients can be computed by using Galerkin method, which states that the best approximation of $v(t,\xi)$ is obtained when the error $\Delta(t,\xi)$, which is defined as

$$\Delta(t,\xi) = Gv(t) + C\frac{dv(t)}{dt} - I(t,\xi(\theta)) \tag{10}$$

is orthogonal to the approximation. That is

$$< \Delta(t,\xi), H_k(\xi) >= 0, \ i = 0, 1, ..., P \tag{11}$$

In this way, we transform the stochastic analysis process to a deterministic process, where we only need to compute the coefficients of its Hermite PC. Once we obtain those coefficients, the mean and variance of the random variables can be easily computed as shown later in the section.

For illustration purpose, we consider one Gaussian variable $\xi = [\xi_1]$ and assume that the node voltage response can be written as second order ($p = 2$) Hermite PC:

$$v(t,\xi) = v_0(t) + v_1(t)\xi_1 + v_2(t)(\xi_1^2 - 1) \tag{12}$$

assuming that the input leakage current sources can also be represented by a second Hermite PC:

$$I(t,\xi) = I_0(t) + I_1(t)\xi_1 + I_2(t)(\xi_1^2 - 1) \tag{13}$$

By applying the Galerkin equation (11) and the orthogonal property of the various order of Hermite PCs, we end up with the following equations

$$Gv_i(t) + C\frac{dv_i(t)}{dt} = I_i(t) \tag{14}$$

where $i = 0, 1, 2, .., P$.

For two independent Gaussian variables, we have

$$\begin{aligned} v(t,\xi) &= v_0(t) + v_1(t)\xi_1 + v_2(t)\xi_2 + v_3(t)(\xi_1^2 - 1) \\ &\quad + v_4(t)(\xi_2^2 - 1) + v_5(\xi_1\xi_2) \end{aligned} \tag{15}$$

Assuming that we have a similar second order Hermite PC for input leakage current $I(t,\xi)$,

$$\begin{aligned} I(t,\xi) &= I_0(t) + I_1(t)\xi_1 + I_2(t)\xi_2 + I_3(t)(\xi_1^2 - 1) \\ &\quad + I_4(t)(\xi_2^2 - 1) + I_5(\xi_1\xi_2) \end{aligned} \tag{16}$$

The (14) is valid with $i = 0, ..., 5$. For more (more than two) Gaussian variables, we can obtained the similar results with more coefficients of Hermite PCs to be solved by using (14).

Once we obtain the Hermite PC of $v(t,\xi)$, we can obtain the mean and variance of $v(t,\xi)$ trivially as (one Gaussian variable case):

$$\begin{aligned} E(v(t,\xi)) &= v_0(t) \\ Var(v(t,\xi)) &= v_1^2(t)Var(\xi_1) + v_2^2(t)Var(\xi_1^2 - 1) \\ &= v_1^2(t) + 2v_2^2(t) \end{aligned} \tag{17}$$

One critical problem remains so far is how to obtain the Hermite PC (13) for leakage current with log-normal distribution. This will be explained in details in the next section.

IV. Hermite PCs for Log-normal Leakage Current Variations

In this section, we present the new method for representing the log-normal leakage current distributions by using Hermite PCs with one or more independent Gaussian variables representing the channel length or threshold voltage variations. Our method is based on [5] and we will show how it can be applied to solve our problems for one or more independent Gaussian variables.

A. Hermite PC representation of log-normal variables

Let $g(\xi)$ be the Gaussian random variable, denoting threshold voltage or device channel length. Let $l(\xi)$ be the random variable obtained by taking the exponential of $g(\xi)$

$$l(\xi) = e^{g(\xi)}, \ g(\xi) = ln(l(\xi)) \tag{18}$$

Obviously, for the MOS device leakage current equation (1), leakage current, $I_{off} = cI_l(V_{th}) = ce^{-V_{th}}$, where the leakage component $I_l(V_{th})$ is a log-normal random variable. Let the mean and the variance of $g(\xi)$ as μ_g and σ_g^2, then the mean and variance of $l(\xi)$ are

$$\mu_l = e^{(\mu_g + \frac{\sigma_g^2}{2})} \tag{19}$$

$$\sigma_l^2 = e^{(2\mu_g + \sigma_g^2)}[e^{\sigma_g^2} - 1] \tag{20}$$

respectively. For a general Gaussian variable $g(\mathbf{x})$, it can always be represented as

$$g(\mathbf{x}) = \sum_{i=0}^{n} \xi_i g_i \tag{21}$$

where ξ_i are orthonormal Gaussian variables. i.e. $< \xi_i, \xi_j >= \delta_{ij}$, $< \xi_i >= 0$ and $\xi_0 = 1$. Note that such form can always

be obtained by using Karhunen-Loeve orthogonal expansion method [6]. In our problem, we need to represent the log-normal random variable $l(\xi)$ by using the Hermite PC expansion form:

$$l(\xi) = \sum_{k=0}^{P} l_k H_k^n(\xi) \tag{22}$$

where $l_0 = \exp[\mu_g + \frac{\sigma_g^2}{2}]$. To find the other coefficients, we can apply (9) on $l(\xi)$. Therefore, we have

$$l_k(t) = \frac{< l(t,\xi), H_k(\xi) >}{< H_k^2(\xi) >}, \ \forall k \in \{0, ..., P\}. \tag{23}$$

It was shown in [5], $l(\xi)$ can be written as

$$l(\xi) = \frac{< H_k(\xi - \mathbf{g}) >}{< H_i^2(\xi) >} = \exp[\mu_g + \frac{1}{2}\sum_{j=1}^{n} g_j^2] \tag{24}$$

where n is the number of independent Gaussian random variables.

The log-normal process can then be written as

$$l(\xi) = l_0(1 + \sum_{i=1}^{n} \xi_i g_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(\xi_i\xi_j - \delta_{ij})}{< (\xi_i\xi_j - \delta_{ij})^2 >} g_i g_j + ...) \tag{25}$$

where g_i is defined in (21).

Next, we show the case of two random variables ($n = 2$). Assume that $\xi = [\xi_1, \xi_2]$ is a normalized uncorrelated Gaussian random variable vector that represents random variable $g(\xi)$:

$$g(\xi) = \mu_g + \sigma_1\xi_1 + \sigma_2\xi_2 \tag{26}$$

Note that

$$< (\xi_i\xi_j - \delta_{ij})^2 > = < \xi_i^2\xi_j^2 > = < \xi_i^2 >< \xi_j^2 > = 1$$

Therefore, the expansion of the log-normal random variables using second order Hermite PCs can be expressed as

$$\begin{aligned} l(\xi) \quad = \quad & l_0(1 + \sigma_1\xi_1 + \sigma_2\xi_2 + \frac{\sigma_1^2}{2}(\xi_1^2 - 1) + \frac{\sigma_2^2}{2}(\xi_2^2 - 1) + \\ & 2\sigma_1\sigma_2\xi_1\xi_2) \end{aligned} \tag{27}$$

where

$$\mu_l = l_0 = \exp(\mu_g + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2)$$

Hence, the desired Hermite PC coefficients, $I_{0,1,2,3,4,5}$, can be expressed as $l_0, l_0\sigma_1, l_0\sigma_2, \frac{1}{2}l_0\sigma_1^2, \frac{1}{2}l_0\sigma_2^2$, and $2l_0\sigma_1\sigma_2$ respectively. Similarly, for four Gaussian random variables, assume that $\xi = [\xi_1, \xi_2, \xi_3, \xi_4]$ is a normalized, uncorrelated Gaussian random variable vector. The random variable $g(\xi)$ can be expressed as

$$g = \mu_g + \sum_{i=1}^{4} \sigma_i\xi_i \tag{28}$$

As a result, the log-normal random variable $l(\xi)$ can be expressed as

$$l(\xi) = l_0(1 + \sum_{i=1}^{4} \xi_i\sigma_i + \sum_{i=1}^{4} \frac{1}{2}(\xi_i^2 - 1)\sigma_i^2 + \sum_{i=1}^{4}\sum_{j=1}^{4} \xi_i\xi_j\sigma_i\sigma_j + ...) \tag{29}$$

where

$$\mu_l = l_0 = \exp(\sigma_0 + \frac{1}{2}\sum_{i=1}^{4} \sigma_i^2)$$

Hence, the desired Hermite PC coefficients can be expressed using the equation (29) above. Once we have the Hermite PC representation of the leakage current sources $I(t, \xi)$, the node voltages $v(t, \xi)$ can be computed by using equations (14) with proper order p of the PCs to obtain all the Hermite PC coefficients of $v(t, \xi)$.

V. Spatial correlation

In this section, we consider the spatial correlation among different variations within a die. Spatial correlations exist in the intra-dia variations in different forms and have been modeled for timing analysis [10], [1]. The general way to consider spatial correlation is by means of mapping the correlated random variables into a set of independent variables. This can be done by using some orthogonal mapping techniques, such as principal component analysis(PCA). In this paper, we also apply PCA method in our spectral statistical analysis framework for power/grid statistical analysis.

A. Concept of Principal Component Analysis

We first briefly review the concept of principal component analysis, which is used here to transform the random variables with correlation to uncorrelated random variables [8].

Suppose that x is a vector of n random variables, $x = [x_1, x_2, ..., x_n]^T$, with covariance matrix C and mean vector $\mu_x = [\mu_{x_1}, \mu_{x_2}, ..., \mu_{x_n}]$. To find the orthogonal random variables, we first calculate the eigenvalue and corresponding eigenvector. Then, by ordering the eigenvectors in descending order eigenvalues, the orthogonal matrix A will be obtained. Here, A is expressed as

$$A = [e_1^T, e_2^T, ..., e_n^T]^T \tag{30}$$

where e_i is the corresponding eigenvector to eigenvalue λ_i, which satisfies

$$\lambda_i e_i = Ce_i, \lambda_i < \lambda_{i-1}, i = 1, 2, ..., n \tag{31}$$

With A, we can perform the transformation to get orthogonal random variables y, $y = [y_1, y_2, ..., y_n]^T$ by using

$$y = A(x - \mu_x) \tag{32}$$

where, y_i is a random variable with Gaussian distribution. The mean, μ_{y_i}, is 0 and the standard deviation, σ_{y_i}, is $\sqrt{\lambda_i}$ on the condition that [8]

$$e_i^T e_i = 1, i = 1, 2, ..., n \tag{33}$$

Here, due to the orthogonal property of matrix A

$$A^{-1} = A^T \tag{34}$$

To reconstruct the original random variables, we use the following equation:

$$x = A^T y + \mu_x \tag{35}$$

B. Spatial Correlation In Statistical Power Grid Analysis

To consider intra-die variation in V_{th}, the chip is divided into n regions. Assuming $\Phi = [\Phi_1, \Phi_2, ..., \Phi_n]$ is a random variable vector, representing the variation of V_{th} on different part of the circuit. In other words, in the *i*th region, the leakage current $I_{off_i} = ce^{V_{th}(\Phi_i)}$, follows the log-normal distribution. Here, Φ_i is a random variable with Gaussian distribution. $\mu_\Phi = [\mu_{\Phi_1}, \mu_{\Phi_2}, ..., \mu_{\Phi_n}]$ is the mean vector of Φ and C is the covariance matrix of Φ.

With PCA, we can get the corresponding uncorrelated random variables $\phi = [\phi_1, \phi_2, ..., \phi_n]$ from the equation

$$\phi = A(\Phi - \mu_\Phi) \tag{36}$$

Also, the original random variables can be expressed as

$$\Phi_i = \sum_{j=1}^{n} a_{ij}\phi_j + \mu_{\Phi_i}, i = 1, 2, ...n \tag{37}$$

where a_{ij} is the *i*th row, *j*th column element in the orthogonal mapping matrix defined in (32). $\phi = [\phi_1, \phi_2, ..., \phi_n]$ is a vector with orthogonal Gaussian random variables. The mean of ϕ_j is 0 and variance is λ_j, $j = 1, 2, ..., n$. The distribution of ϕ_i can be written as

$$\phi_i = \mu_{\phi_i} + \sigma_{\phi_i}\hat{\xi}_i, i = 1, 2, ..., n \tag{38}$$

$\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n]$ is a vector with orthogonal normal Gaussian random variables. Φ_i can be expressed with normal random variables, $\hat{\xi} = [\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_n]$:

$$\Phi_i = \sum_{j=1}^{n} a_{ij}\sqrt{\lambda_j}\hat{\xi}_j + \mu_{\Phi_i}, i = 1, 2, ..., n \tag{39}$$

With (39), the leakage current can be expanded as Hermite Polynomial Chaos:

$$I(\Phi_i) \sim e^{\Phi_i} = e^{\sum_{j=1}^{n} g_j\hat{\xi}_j + \mu_{\Phi_i}} = \mu_i(1 + \sum_{j=1}^{n}\hat{\xi}_j g_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{(\hat{\xi}_j\hat{\xi}_k - \delta_{jk})}{<(\hat{\xi}_j\hat{\xi}_k - \delta_{jk})^2>}g_j g_k + ...) \tag{40}$$

Here,

$$g_j = a_{ij}\sqrt{\lambda_j}, j = 1, 2, ..., n \tag{41}$$

Therefore, the MNA equation with correlated random variables Φ in current source can be expressed in terms of uncorrelated random variables $\hat{\xi}$ as follows:

$$Gv(t) + C\frac{dv(t)}{dt} = I_i(t, \hat{\xi}) \tag{42}$$

With orthogonal property of $\hat{\xi}$, (42) will simply be solved by using (14), $i = 1, 2, ..., P$.

VI. Experimental Results

This section describes the simulation results of circuits with log-normal leakage current distributions for a number of power grid networks. All the proposed methods have been implemented in Matlab. All the experimental results are carried out in a Linux system with dual Intel Xeon CPUs with 3.06Ghz and 1GB memory.

TABLE I
ACCURACY COMPARISON BETWEEN HERMITE PC (HPC) AND TAYLOR EXPANSION

δ_g	0.01	0.1	0.3	0.5	0.7
HPC (%)	3.19	1.88	2.07	5.5	2.92
Taylor (%)	3.19	1.37	2.41	16.6	24.02

A. Comparison with Taylor expansion method

We first compare the proposed method with the simple Taylor expansion method for one and more Gaussian variables. For simplicity, we assume one Gaussian random variable $g(\xi)$, which is expressed as

$$g = \mu_g + \sigma_g\xi \tag{43}$$

where ξ is a normalized Gaussian random variable with $<\xi>$ = 0, and $<\xi^2>$ = 1. The log-normal random variable $l(\xi)$, obtained from $g(\xi)$, is written as

$$l(\xi) = e^{g(\xi)} = \exp(\mu_g + \sigma_g\xi) \tag{44}$$

Expand the exponential into Taylor series and keep all the terms up to second order, then we have

$$\begin{aligned} l(\xi) &= 1 + \sum_{i=0}^{1}\xi_i g_i + \frac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}\xi_i\xi_j g_i g_j + ... \\ &= 1 + \mu_g + \frac{1}{2}\mu_g^2 + \frac{1}{2}\sigma_g^2 + (\sigma_g + \mu_g\sigma_g)\xi + \\ &\quad \frac{1}{2}\sigma_g^2(\xi^2 - 1) + ... \end{aligned} \tag{45}$$

We observe that the second-order Taylor expansion, as shown in (45), is similar to second order Hermite PC in (27). Hence, the Galerkin method can still be applied, we then use (14) to obtain the Hermite PC coefficients of node voltage $v(t, \xi)$ accordingly. We want to emphasize, however, that the polynomials generated by Taylor expansion in general are not orthogonal with respect to Gaussian distributions and can't be used with Galerkin method, unless we only keep the first order of Taylor expansion results (with less accuracy). In this case, the resulting node voltage distribution is still Gaussian, which obviously is not correct. We note that the first order Taylor expansion has been used in the statistical timing analysis [1]. The delay variations, due to interconnects and devices, can be approximated with this limitation. The skew distributions may be computed easily with Gaussian process.

To compare these two methods, we use the Monte Carlo method as to measure the accuracies of two methods in terms of standard deviation. For Monte Carlo, we sample 2000 times and the results are summarized in Table I. In this table, δ_g is the standard deviation of the Gaussian random threshold voltage Gaussian variable in the log-normal current source. *HPC* is the standard deviation from the Hermite PC method in terms of relative percentage against the MC method. *Taylor* is the standard deviation from the Taylor expansion method in terms of relative percentage against the MC method. We can observe that when the variation of current source increases, the Taylor expansion method will result in significant errors compared to the MC method, while the proposed method

TABLE II

CPU TIME COMPARISON WITH THE MONTE CARLO METHOD

Ckt	#node	p	n	MC(s)	HPC(s)	Speedup
gridrc_3	33	2	1	5.39	0.006	898.33
gridrc_10	1720	2	1	60506.25	61.48	984.16
gridrc_12	3024	2	2	3.13×10^5	625.63	499.61
gridrc_67	7400	2	2	N/A	1979	N/A

has the smaller errors for all cases. This clearly shows the advantage of the proposed method.

B. Examples without Spatial Correlation

In the case of two random variables with one large and the other small standard deviations, the larger one dominates, which shows the shape of log-normal as in Fig. 1.

To consider multiple random variables, we divide the circuit into several partitions. We first divide the circuit into two parts. Fig. 1 shows the node voltage of one node of a particular time instance of a ground network with 336 nodes with two independent variables. The standard deviations for two Gaussian variations are $\sigma_{g1} = 0.5$, $\sigma_{g2} = 0.1$. The 3δ variations are also marked in the figure. Table II shows the

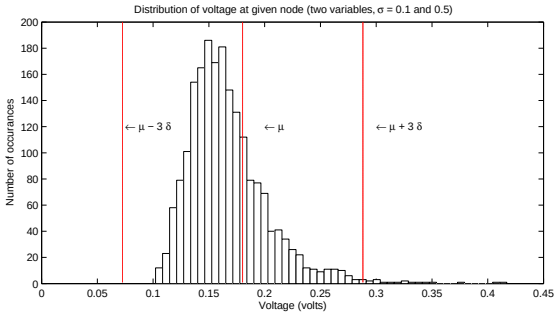

Fig. 1. Distribution of the voltage in a given node with two Gaussian variables, $\sigma_{g1} = 0.1$ and $\sigma_{g2} = 0.5$

speedup of the Hermite PC method over Monte Carlo method with 3000 samples. In this table, *#node* is the number of nodes in the power grid circuits. p is the order of the Hermite PCs and n is the number of independent Gaussian random variables. HPC and MC represent the CPU times in seconds used for Hermite PC and MC method respectively. It can be seen that the proposed method is about two order of magnitude faster than the MC method. When more Gaussian variables are used for modeling intra-die variations, we need more Hermite PC coefficients to compute. Hence, the speedup will be smaller if the MC method uses the same number of samples as shown in *gridrc_12*. Also, one observation is that the speedup depends on the sampling size in MC method. We found that 2000 to 3000 samples are the reasonable numbers to have good MC results. Note that the large-sized circuit, such as *gridrc_67*, is unable to finish within reasonable time using MC. The advantage of HPC is obvious as shown in the table.

C. Examples with Spatial Correlation

To model the intra-die variations with spatial correlations, we divide the power grid circuit into several parts. We first

TABLE III

COMPARISON BETWEEN NON-PCA AND PCA AGAINST MONTE CARLO METHODS

ckt	#nodes	Mean		Std Dev	
		Non-PCA % error	PCA % error	Non-PCA % error	PCA % error
1	336	10.3	0.52	18.8	1.13
2	645	8.27	0.59	11.4	1.16
3	1160	10.8	0.50	2.6	0.73

consider that circuit is partitioned into two parts. In this case, we have two independent random current variables, ξ_1 and ξ_2. The correlated variables for the two parts are $\Phi_1 = \xi_1 + 0.5\xi_2$ and $\Phi_2 = \xi_2 + 0.5\xi_1$ respectively as shown in Fig. 2. Table III shows the error percentage of mean and standard

$\Phi_1 = \xi_1 + 0.5\xi_2$	$\Phi_2 = \xi_2 + 0.5\xi_1$

Fig. 2. Correlated random variables setup in ground circuit devided into two parts

deviation of the comparison between *Monte Carlo* and *HPC* with Principal Component Analysis(PCA) and the comparison between *Monte Carlo* and *HPC* without PCA. As shown in the table, it is necessary to use PCA when spatial correlation is considered. Fig.3 shows the node voltage distribution of one certain node in a ground network with 336 nodes, using both PCA and non-PCA method. To get more accuracy, we divide

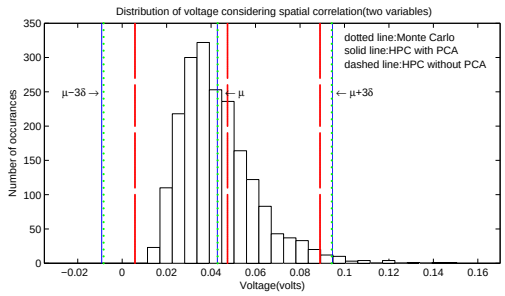

Fig. 3. Distribution of the voltage in a given node with two Gaussian variables with spatial correlation

ϕ1=ζ1+0.5ζ2+0.5ζ3	ϕ3=ζ3+0.5ζ1+0.5ζ4
ϕ2=ζ2+0.5ζ1+0.5ζ4	ϕ4=ζ4+0.5ζ2+0.5ζ3

Fig. 4. Correlated random variables setup in ground circuit divided into four parts

the circuit into four parts and each part has correlation with its neighbor as shown in Fig.4. ϕ is the correlated random variable vector we use in the circuit. $\zeta = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]$ are independent Gaussian distribution random variables with standard deviations $\zeta_1 = 0.1$, $\zeta_2 = 0.2$, $\zeta_3 = 0.1$ and $\zeta_4 = 0.5$. Fig.5 is the voltage distribution of a given node. The mean voltage and voltages of worst case are given as the blue line. Note that the size of the ground networks we analyzed is

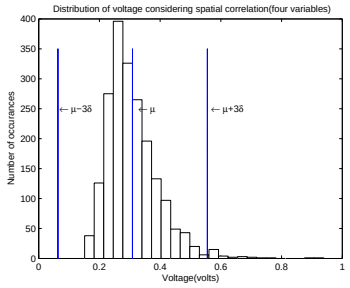

Fig. 5. Distribution of the voltage in a given node with four Gaussian variables with spatial correlation

mainly limited by the solving capacity of Matlab on a single Intel CPU Linux workstation. Given long simulation time due to large Monte Carlo sampling runs, we limit the ground network size to about 3000 nodes.

Also note that for more accurate modeling, we need to have more partitions of the circuits and thus more independent Gaussian variables are needed as shown in [1].

VII. Conclusion

In this paper, we have proposed a new stochastic simulation method for fast estimating the voltage variations due to the process-induced log-normal leakage current variations with spatial correlations. The new analysis is based on the Hermite polynomial chaos (PC) representation of random processes. We extended the existing Hermite PC based power grid analysis method [7] by considering log-normal leakage distribution with spatial correlations, instead of Gaussian current distribution without spatial correlations. Our experimental results show that the new method is more accurate than the Gaussian-only Hermite PC using the Taylor expansion method for analyzing leakage current variations and two orders of magnitude faster than Monte Carlo methods with small variation errors. In the presence of spatial correlations, methods without considering the spatial correlations may lead to large errors, roughly 8%-10% in our tested cases. However, our proposed method leads to about 1% or less of errors in both mean and standard deviations.

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