Generating Stable and Sparse Reluctance/Inductance Matrix under Insufficient Conditions

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***Abstract*— This paper presents generating stable and sparse reluctance/inductance matrix from the inductance matrix which is extracted under insufficient discretization. So far, to generate the sparse reluctance matrix with guaranteed stability, this matrix has to be diagonally dominant M matrix. Hence, the repeated inductance extractions are necessary using a smaller grid size, in order to obtain the well-defined matrix. Alternatively, this paper provides some ideas for generating the sparse reluctance matrix, even if the extracted reluctance matrix is not diagonally dominant M matrix, precisely, the positive off-diagonal elements are even found. This eases the extraction tasks greatly. Furthermore, the sparse inductance matrix is also generated by using the practical and sophisticated double inverse methods, which is useful for the SPICE simulation, since reluctance components are not still supported in SPICE-like simulators.**

I. Introduction

With the progress of VLSI technologies, accurate modeling of interconnects has become increasingly important. To overcome the signal/power integrity issues of the VLSI systems, the timing analysis of on-chip interconnects including package and PCB is necessary. The Partial Element Equivalent Circuit (PEEC) model has been widely used for the interconnect modeling. However, the inductance/capacitance matrix is dense which needs unrealistic CPU time and memory for the circuit simulation. To carry out the timing analysis practically, generating a sparse inductance matrix is required, and several approaches have been reported in [1], [2], [3].

While truncating the off-diagonal terms of the inductance matrix simply, the positive definiteness of the inductance matrix may not be preserved, which causes unstable circuit simulation. The shift truncate [3] and the double inverse [1], [2] methods guarantee the stability of the truncated inductance matrix, but these methods sacrifice accuracy of simulation. Devgan *et al.* suggested that the inverse of the inductance matrix, i.e., the reluctance matrix, preserves the positive definiteness after the truncation [4]. However, Chen *et al.* [5] showed that the application is restricted to the case that interconnects are finely discretized, more precisely, when the reluctance matrix is diagonally dominant M matrix [6]. Therefore, the inductance extraction has to be repeated with a small gid size until the extracted reluctance matrix becomes diagonally dominant M matrix.

In this paper, we present the techniques for generating stable and sparse reluctance/inductance matrix from the inductance matrix which is extracted under insufficient discretization. In [5], the inductance extractions are repeated using a small grid size until the positive off-diagonal terms of the reluctance matrix have not been found. However, even if such a reluctance matrix is found, this matrix may be extremely large scale, which needs large CPU time for circuit simulation. Alternatively, we treat the reluctance matrix directly even if it has positive off-diagonal elements, and the off-diagonal elements which are smaller in magnitude than a threshold, are truncated. However, since we can not say here positive definite of the truncated reluctance matrix, the Cholesky decomposition is applied to the truncated reluctance matrix in order to confirm the positive definiteness rather than solving linear equations. Since the reluctance matrix, which is obtained by the inductance matrix extracted under insufficient discretization, has nearly the same structure as one produced under sufficient discretization, the direct truncating does not fail in our many experiments. Therefore, the Cholesky decomposition is used as an assurance and it would be essentially unnecessary, while the decomposition is efficiently done since the reluctance matrix is always sparse.

In addition to the direct truncation of the reluctance matrix, a method for enforcing the diagonally dominance is given. Although this method is worse in accuracy than the direct truncating, it has an advantage that confirmation of stability is not necessary. Moreover, direct truncating is also used in the double inverse method [1], [2] by defining a proper threshold, in which case loss of accuracy in circuit simulation is prohibited differently from [1], [2]. Furthermore, combing with the Neumann series [7], we generate a sparse inductance matrix which has the same structure as the truncated reluctance matrix. Although our attention is to generate a sparse inductance matrix under insufficient conditions, this idea is also applicable to when conductors are sufficiently discretized [5].

The proposed methods are applied to a PCB. The performance of SPICE transient analysis is evaluated for the system including the PCB and package. Even though the extraction is done under insufficient discretization, good SPICE simulation results are obtained both in accuracy and efficiency.

II. Reluctance Matrix

Reluctance matrix $\boldsymbol{K}$ is defined as the inversion of an inductance matrix $\boldsymbol{L}$. Therefore, we must first extract the inductance matrix for obtaining the reluctance one. In our approach, the concept of partial inductance matrix is used, where the inductance matrix is efficiently extracted since the explicit information about the current return paths is not necessary [3], [5].

Element of the partial inductance matrix is given by

$$\mathcal{L}_{ij} = \frac{1}{I_j a_i}\left[\int_{a_i}\int_{\vec{l}_i} \vec{A}_{ij}\cdot d\vec{l}_i da_i\right]. \qquad (1)$$

In this expression, $\vec{A}_{ij}$ is the magnetic vector potential along segment i due to the current $\vec{I}_j$ in segment j. Segment i has a cross section a_i. Then, the inductively coupled conductor system is written by

$$\boldsymbol{L}\boldsymbol{I} = \boldsymbol{\Phi}, \qquad (2)$$

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where

$$
\begin{aligned}
\boldsymbol{L} &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} \\ \vdots & & \ddots & \vdots \\ \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} \end{bmatrix}, \quad (3) \\
\boldsymbol{\Phi} &= \begin{bmatrix} \sum_{j=1}^{n} \frac{1}{a_1} \left[\int_{a_1} \int_{\vec{I}_1} \vec{A}_{1j} \cdot d\vec{I}_1 da_1 \right] \\ \vdots \\ \vdots \\ \sum_{j=1}^{n} \frac{1}{a_n} \left[\int_{a_n} \int_{\vec{I}_n} \vec{A}_{nj} \cdot d\vec{I}_n da_n \right] \end{bmatrix}, \quad (4) \\
\boldsymbol{I} &= [I_1, I_2, \cdots, I_n]^T. \quad (5)
\end{aligned}
$$

The reluctance matrix is given by $\boldsymbol{K} = \boldsymbol{L}^{-1}$. Elements of the reluctance matrix are written by

$$
\boldsymbol{K} = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \cdots & \mathcal{K}_{1n} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \cdots & \mathcal{K}_{2n} \\ \vdots & & \ddots & \vdots \\ \mathcal{K}_{n1} & \mathcal{K}_{n2} & \cdots & \mathcal{K}_{nn} \end{bmatrix}. \quad (6)
$$

In [5], it is shown that the reluctance matrix becomes symmetric positive definite and diagonally dominant, if the conductors are finely discretized. Moreover, $\mathcal{K}_{ii} > 0$ and $\mathcal{K}_{ij} < 0$ for $i \neq j$ is shown, which implies the reluctance matrix is M matrix [6]. Therefore, the sparse reluctance matrix can be easily obtained from truncating the small off-diagonal terms preserving the positive definiteness. These useful properties of the reluctance matrix are given when conductors are finely discretized. However, this does not mean that the inductance and reluctance matrices become unstable unless the conductors are finely discretized. The inductance and reluctance matrices are always positive definite [8]. Roughly speaking, since the energy $\boldsymbol{I}^T \boldsymbol{L} \boldsymbol{I}$ of the linear system of (2) is always positive, the partial inductance matrix $\boldsymbol{L}$ is positive definite, and then the reluctance matrix $\boldsymbol{K}$ becomes also positive definite.

In [5], an example of the inductance matrix is given, where conductors are roughly discretized. The inductance matrix is given by

$$
\boldsymbol{L}_5 = \begin{bmatrix} 1.04 & 0.34 & 0.37 & 0.24 & 0.51 \\ 0.34 & 0.45 & 0.09 & 0.06 & 0.27 \\ 0.37 & 0.09 & 1.04 & 0.34 & 0.41 \\ 0.24 & 0.06 & 0.34 & 0.45 & 0.11 \\ 0.51 & 0.27 & 0.41 & 0.11 & 1.69 \end{bmatrix} \times 10^{-10} H,
$$

and the reluctance matrix is given by

$$
\boldsymbol{K}_5 = \begin{bmatrix} 1.57 & -0.94 & -0.22 & -0.47 & -0.25 \\ -0.94 & 3.02 & 0.15 & 0.01 & -0.23 \\ -0.22 & 0.15 & 1.42 & -0.93 & -0.24 \\ -0.47 & 0.01 & -0.93 & 3.12 & 0.16 \\ -0.25 & -0.23 & -0.24 & 0.16 & 0.75 \end{bmatrix} \times 10^{10} H^{-1}.
$$

Some of off-diagonal elements of $\boldsymbol{K}_5$ are positive and the matrix is not diagonally dominant. However, $\boldsymbol{K}_5$ has only positive eigenvalues: $\{0.4217, 1.0000, 1.3539, 3.4503, 3.6996\} \times 10^{-9}$. This example shows that the reluctance and inductance matrices are positive definite even if $\mathcal{K}_{ii} > 0$ and $\mathcal{K}_{ij} < 0$ for $i \neq j$ is not satisfied.

When a reluctance matrix is symmetric and diagonally dominant, (that is, positive definite), after truncating (i,j) and (j,i) elements for $i \neq j$, the resultant matrix is still diagonally dominant. Therefore, stable and sparse reluctance matrix is easily obtained by truncating the off-diagonal terms simply. However, is it impossible to truncate the reluctance matrix which is not diagonally dominant? Discard the four off-diagonal elements such as

$$
\tilde{\boldsymbol{K}}_5 = \begin{bmatrix} 1.57 & -0.94 & -0.22 & -0.47 & -0.25 \\ -0.94 & 3.02 & 0.00 & 0.00 & -0.23 \\ -0.22 & 0.00 & 1.42 & -0.93 & -0.24 \\ -0.47 & 0.00 & -0.93 & 3.12 & 0.16 \\ -0.25 & -0.23 & -0.24 & 0.16 & 0.75 \end{bmatrix} \times 10^{10} H^{-1}.
$$

The eigenvalues of $\tilde{\boldsymbol{K}}_5$ are $\{0.3861, 0.9903, 1.4077, 3.3991, 3.6967\} \times 10^{-9}$ which are not only all positive but also near to the eigenvalues of $\boldsymbol{K}_5$. From this fact, we cay say that the sparse reluctance matrix might be obtained by discarding the off-diagonal terms even if inductance extraction is done under insufficient discretization. In [5], when a positive off-diagonal element is found, the extraction result is spoiled and a new extraction is done using finer segments. However, for a realistic physical structure, fulfilling being diagonally dominant is not necessarily easy. Then, we need the repeated extractions so that it is time consuming. Hence, we provide the techniques for generating the sparse reluctance matrix from the matrix which is neither symmetric diagonally dominant nor M matrix.

III. Sparse Reluctance Matrix

A. Direct Truncation

The sparse reluctance matrix is obtained by defining a threshold h_K and discarding the off-diagonal terms of the reluctance matrix (6) if $|\mathcal{K}_{ij}| < h_K$ for $i \neq j$. However, since the positive definiteness then is not guaranteed, we must confirm it for stable circuit simulation. This procedure is very simple. Writing the sparse matrix as $\tilde{\mathcal{K}}$, we apply the Cholesky decomposition to it. Then, if $\tilde{\mathcal{K}} = \boldsymbol{R}^T \boldsymbol{R}$ is given, $\tilde{\mathcal{K}}$ is positive definite. Since the matrix $\tilde{\mathcal{K}}$ is sparse, the Cholesky decomposition is efficiently done. Thus, we can easily confirm the positive definiteness of a truncated reluctance matrix.

A property of eigenvalues is given from the following lemma:

Lemma 1 ([6]): If $\boldsymbol{A}$ and $\boldsymbol{A}+\boldsymbol{E}$ are n-by-n symmetric matrices, then,

$$
\lambda_k(\boldsymbol{A}) + \lambda_n(\boldsymbol{E}) \leq \lambda_k(\boldsymbol{A}+\boldsymbol{E}) \leq \lambda_k(\boldsymbol{A}) + \lambda_1(\boldsymbol{E}) \qquad k = 1, \ldots, n \quad (7)
$$

where $\lambda_k(\boldsymbol{A})$ is the k-th largest eigenvalues, thus, $\lambda_n(\boldsymbol{A}) \leq \cdots \lambda_2(\boldsymbol{A}) \leq \lambda_1(\boldsymbol{A})$.

Define the matrix $\boldsymbol{E}$ as it has zero elements except for (i,j) and (j,i) elements which are equal to the corresponding elements of $\boldsymbol{A}$. The eigenvalues of $\boldsymbol{E}$ are 0's and $\pm|a_{ij}|$, where $|a_{ij}|$ is magnitude of the (i,j) element. From the **Lemma 1**, the eigenvalues of the discarded matrix is bounded: $\lambda_k(\boldsymbol{A}) - |a_{ij}| \leq \lambda_k(\boldsymbol{A}-\boldsymbol{E}) \leq \lambda_k(\boldsymbol{A}) + |a_{ij}|$, $k = 1, 2, \ldots, n$, which says that a small off-diagonal element does not have large influence on the eigenvalues. This property is used in all the truncation techniques implicitly. As first suggested in [4], reluctance has the shielding effect and decreases more greatly with increase of distance between two conductors than inductance. Therefore, we can drop many off-diagonal terms of the reluctance matrix without large changes of eigenvalues. Generally, the positive off-diagonal terms are found having small or relative small magnitude as $\boldsymbol{K}_5$ in Sect. II. Therefore, even though the small off-diagonal terms are discarded, the possibility of the reluctance matrix being positive definite is high. In our many examples, the Cholesky decomposition never fails, which means that the truncated reluctance matrix is still positive definite. Therefore, stable and sparse reluctance matrix is almost always given by discarding small off-diagonal terms and the Cholesky decomposition is merely used as an assurance.

B. Enforcing Positive Definiteness

To drop the off-diagonal terms enforcing the positive definiteness of the reluctance matrix, the same idea as in the double inverse method [1], [2] for truncating the inductance matrix is used. Define the matrix $\boldsymbol{E}$ as $-|a_{ij}|$ for (i,i) and (j,j) and a_{ij} for (i,j) and (j,i), otherwise zero, where a_{ij} is the (i,j) element of a positive definite matrix $\boldsymbol{A}$ for $i \neq j$. Then, the (i,j) and (j,i) elements of the matrix $\boldsymbol{A}-\boldsymbol{E}$ becomes zero. Since $-\boldsymbol{E}$ is positive definite, sum of two positive definite matrices $\boldsymbol{A}$ and $-\boldsymbol{E}$ are also positive.

C. Enforcing Diagonally Dominance

In contrast to direct truncating in Sect. III.A, enforcing positive definiteness in Sect. III.B has unwanted feature that magnitude of the diagonal elements increases with the number of discarded off-diagonal elements. Even though the inductance extraction is done under insufficient discretization, the reluctance matrix has nearly the same properties as one produced under sufficient discretization. Therefore, the number of positive off-diagonal elements is fewer than the negative ones and magnitude of the positive elements is generally small. Hence, we take advantage of these features of the reluctance matrix.

A symmetric matrix $\boldsymbol{A}$ is decomposed into the three types of matrices, $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, and $\boldsymbol{D}$. $\boldsymbol{E}_1$ has $-a_{ij}$ for (i,i) and (j,j) and $a_{ij} < 0$ for (i,j) and (j,i), otherwise zero. $\boldsymbol{E}_2$ has $-a_{ij}$ for (i,i) and (j,j) and $a_{ij} \geq 0$ for (i,j) and (j,i), otherwise zero. $\boldsymbol{D}$ has the diagonal matrix which is obtained after subtracting all the $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ type matrices from $\boldsymbol{A}$. Moreover, $\boldsymbol{D}$ is divided into the two matrices $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ that have positive and negative values of $\boldsymbol{D}$, respectively. For example, the reluctance matrix $\boldsymbol{K}_5$ in the previous section is represented by

$$\begin{aligned} \boldsymbol{K}_5 \;=\; & \underbrace{\begin{bmatrix} 1.88 & -0.94 & -0.22 & -0.47 & -0.25 \\ -0.94 & 1.17 & 0.00 & 0.00 & -0.23 \\ -0.22 & 0.00 & 1.39 & -0.93 & -0.24 \\ -0.47 & 0.00 & -0.93 & 1.40 & 0.00 \\ -0.25 & -0.23 & -0.24 & 0.00 & 0.72 \end{bmatrix}}_{\text{sum of } \boldsymbol{E}_1 \text{ type matrices}} \\ + & \underbrace{\begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & -0.16 & 0.15 & 0.01 & 0.00 \\ 0.00 & 0.15 & -0.15 & 0.00 & 0.00 \\ 0.00 & 0.01 & 0.00 & -0.17 & 0.16 \\ 0.00 & 0.00 & 0.00 & 0.16 & -0.16 \end{bmatrix}}_{\text{sum of } \boldsymbol{E}_2 \text{ type matrices}} \\ + & \underbrace{\operatorname{diag}(0.00, 2.01, 0.18, 1.89, 0.19)}_{\boldsymbol{D}_1} \\ + & \underbrace{\operatorname{diag}(-0.31, 0.00, 0.00, 0.00, 0.00)}_{\boldsymbol{D_2}} \quad \times 10^{10} H^{-1}. \end{aligned}$$

The sum of $\boldsymbol{D_1}$ and all the $\boldsymbol{E}_1$ type matrices are symmetric and diagonally dominant, i.e., positive definite, if there is at least an element in each row. Hence, after summing $\boldsymbol{D_1}$ and all the $\boldsymbol{E}_1$ type matrices, the sparse matrix is obtained by truncating the off-diagonal terms.

D. SPICE Model of Reluctance Matrix

The SPICE model of reluctance matrix is provided by using a multiple-input voltage controlled voltage source (VCVS) [9]. The equivalent circuit is derived from the reciprocal relation of inductively coupled system in the complex domain: $\boldsymbol{I} = (1/s)\boldsymbol{KV}$, where $\boldsymbol{I}$ and $\boldsymbol{V}$ are branch currents and voltage through self inductors.

The i-th row of $\boldsymbol{I} = (1/s)\boldsymbol{KV}$ is written by

$$\sum_{j=1}^{n} K_{ij} V_j = s I_i. \tag{8}$$

Then, the SPICE model is derived from

$$V_i = s\frac{1}{K_{ii}} I_i - \sum_{j=1, j\neq i}^{n} \frac{K_{ij}}{K_{ii}} V_j, \tag{9}$$

where $1/K_{ii}$ and K_{ij}/K_{ii} represent an inductor and a VCVS, respectively [9].

SPICE uses nonlinear simulation techniques even when all components are linear, which requires solving a set of linear equations at every time point. When a VCVS is used, one circuit variable is added to the MNA equation. Hence, the circuit order grows with the number of mutual couplings if the SPICE model of (9) is used. Therefore, the SPICE simulation using the VCVS's is efficient only when the number of mutual couplings is not large. Therefore, sparse inductance matrix is preferable to the reluctance, and the next section provides generating sparse inductance matrix.

IV. Sparse Inductance Matrix

A. Direct Truncating

After obtaining a sparse reluctance matrix, it is again inverted. The resultant matrix, which is a kind of inductance matrix, can be truncated easier than the original inductance matrix. Since inverse matrix is calculated twice, this method is called the double inverse method [1], [2]. In this method, the sparse inductance matrix is obtained by enforcing positive definiteness as in Sect. III.B. Alternatively, we use the direct truncating, defining a threshold h_L. Next, the Cholesky decomposition is applied in order to confirm the positive definiteness. Differently from the case of the reluctance matrix, the Cholesky decomposition often fails. Hence, we must select a proper h_L for obtaining a positive definite matrix while repeating the Cholesky decompositions. Nevertheless the fault of the repeated decompositions, this approach is beneficial on accuracy since magnitude of diagonal elements is unchanged.

B. Neumann Series

In Sect. III.C, the reluctance matrix is enforced as diagonally dominant. In this case, the inversion is carried out using the Neumann series [7]. When a symmetric matrix $\boldsymbol{A}$ is appropriately split as $\boldsymbol{A} = \boldsymbol{M} - \boldsymbol{N}$, the inverse matrix can be written by

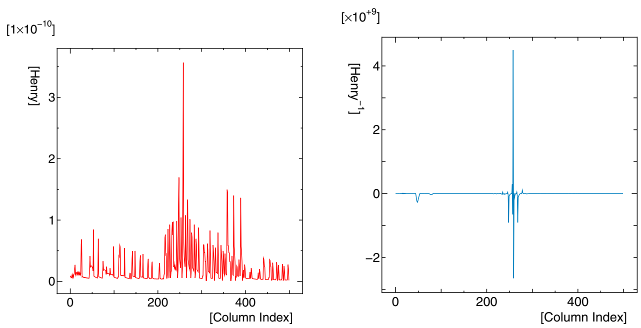
$$\boldsymbol{A}^{-1} \;=\; \left(\sum_{i=0}^{\infty} \left(\boldsymbol{M}^{-1}\boldsymbol{N}\right)^{i} \right) \boldsymbol{M}^{-1}. \tag{10}$$

For the finite series $\boldsymbol{A}_p^{-1} = \left(\sum_{i=0}^{p} \left(\boldsymbol{M}^{-1}\boldsymbol{N}\right)^{i}\right) \boldsymbol{M}^{-1}$ of (10), the following lemma is known.

Lemma 2 ([7]): Let $\boldsymbol{A}$ be a real $n \times n$ symmetric positive definite matrix and let $\boldsymbol{M}^{-1}$ be any real $n \times n$ symmetric positive definite matrix such that $\rho(\boldsymbol{M}^{-1}\boldsymbol{N}) < 1$. Then, for $p > 1$, $\boldsymbol{A}_p^{-1}$ is symmetric and positive definite, where $\rho(\cdot)$ is spectrum radius.

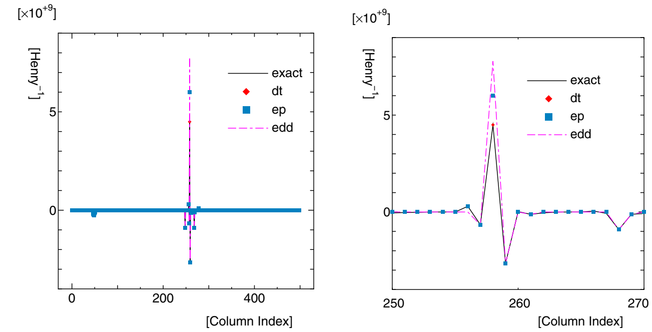
TABLE I
CPU TIME [SEC.] COMPARISON OF SPICE SIMULATIONS

method	threshold1	threshold2	ratio1	ratio2	CPU times
exact					21,108
dt	10^9		0.70	0.69	61
ep	10^9		0.70	0.69	51
edd	10^9		0.68	0.68	53
dt	10^8		2.19	2.01	1,090
ep	10^8		2.19	2.01	1,185
edd	10^8		1.88	1.75	867
dt2	10^8	10^{-10}	0.27	0.48	10
dt2	10^8	10^{-12}	4.12	9.62	847
pinv	10^8		2.10	2.01	175



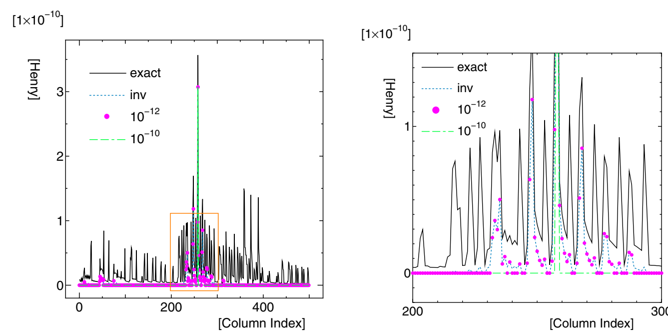
(a) (b)

Fig. 2. The mutual couplings in the 259-th row of the 500×500 inductance and reluctance matrices. (a)Inductance. (b)Reluctance.



(a) (b)

Fig. 3. The mutual couplings in the 259-th row of the truncated 500×500 reluctance matrix. (a)All values. (b)The enlarged figure of 3(a).



(a) (b)

Fig. 4. The mutual couplings in the 259-th row of the truncated 500×500 inductance matrix. (a)All values. (b)The enlarged figure of the rectangular part of 4(a).

After applying the method in Sect. III.C, put the sparse reluctance matrix as $\tilde{\boldsymbol{K}}$ and split it into $\tilde{\boldsymbol{K}} = \boldsymbol{D}_1 - \boldsymbol{N}$. Then, we can obtain the sparse inductance matrix:

$$\tilde{\boldsymbol{L}}_p = \left(\sum_{i=0}^{p} \left(\boldsymbol{D}_1^{-1} \boldsymbol{N} \right)^i \right) \boldsymbol{D}_1^{-1}. \qquad (11)$$

Since absolute sums of all the off-diagonal elements of $\boldsymbol{D}_1^{-1}\boldsymbol{N}$ in each row is less than one, $\rho(\boldsymbol{M}^{-1}\boldsymbol{N}) < 1$ from Gershgolin's theorem [6]. Thus, the inductance matrix $\tilde{\boldsymbol{L}}_p$ is positive definite from **Lemma 2.**

Since $-\boldsymbol{N}$ expresses the off-diagonal terms of the truncated reluctance matrix $\tilde{\boldsymbol{K}}$, the inductance matrix $\tilde{\boldsymbol{L}}_1$ has the same sparse pattern as $\tilde{\boldsymbol{K}}$. However, with a larger p, $\tilde{\boldsymbol{L}}_p$ is not sparse yet. Thus, the first or second order is preferable.

The finite Neumann series (11) reveals why the inductance matrix obtained after truncation of the reluctance matrix is easily truncated as suggested in [1], [2]. The sparser $\boldsymbol{A}$ is, the closer the eigenvalues of $\boldsymbol{D}_1^{-1}\boldsymbol{N}$ distributes around the origin. Then, the finite Neumann series (10) as $\boldsymbol{M} = \boldsymbol{D}_1$ quickly converges, which means that the off-diagonal terms tend to have small magnitude.

V. SIMULATIONS

A. Generating Sparse Matrices

Figure 1 shows the system composed of PCB and package. This was provided by Hitachi. The PCB model consists of two inductance and one capacitance matrices and 4 inductors as via model. The inductance matrices were extracted in the two perpendicular directions separately and the size of these matrices are 500×500 and 493×493. The size of the capacitance matrix is 568×568. The package has 21 ports and the model is composed of inductance and resistance matrices. Figure 2(a) shows the mutual couplings in the 259-th row of the 500×500 inductance matrix. The profile is so complicated that the reluctance matrix can not be truncated preserving the positive definiteness, even if any threshold is used. Figure 2(b) shows the mutual couplings in the reluctance matrix. Since the profile is more simple than the inductance, the reluctance matrix seems to be truncated with guaranteed stability. However, the two reluctance matrices are not diagonally dominant. In the 500×500 and 493×493 reluctance matrices, 7.82 and 7.00 percents of the whole elements were positive off-diagonal elements, respectively.

Figures 3(a) and 3(b) shows the mutual couplings of the sparse reluctance matrix obtained with a threshold $h_K = 10^{-8}$, where 'exact', 'dt', 'ep', and 'edd' are the original matrix, the sparse matrices produced by direct truncating in Sect. III.A, enforcing positive definiteness in Sect. III.B, and enforcing diagonally dominance in Sect. III.C, respectively. The result of 'dt' is identical to 'exact' in Fig. 3(b). Since magnitude of truncated off-diagonal elements is added to the corresponding diagonal elements, the values of Column Index 258 in 'ep' and 'edd', which is corresponding to the $(259, 259)$ element of reluctance matrix, are different from the exact value.

Figures 4(a) and 4(b) shows the mutual couplings of the inductance matrix truncated by direct truncating in Sect IV.A, where the reluctance matrix first was discarded with a threshold 10^8 and its inverse was taken. In Figs. 4(a) and 4(b), 'inv' shows the mutual couplings

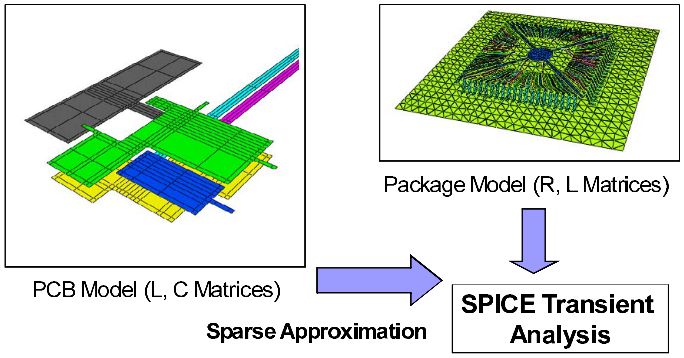
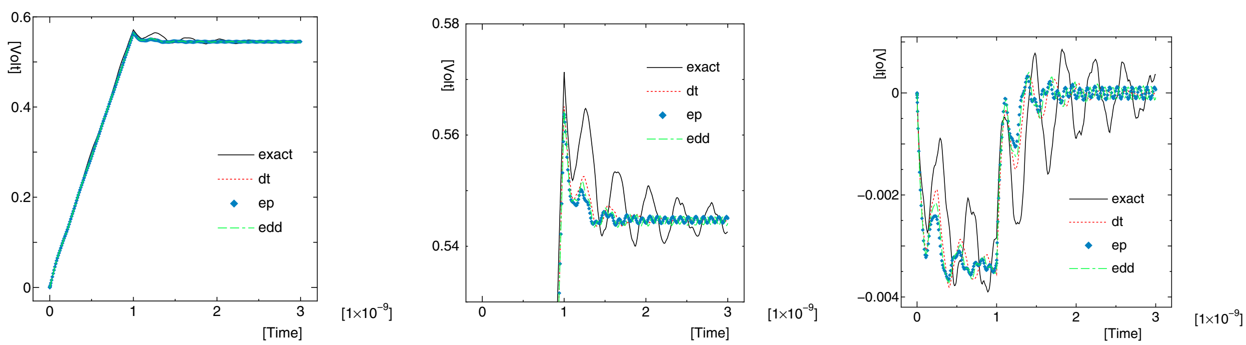
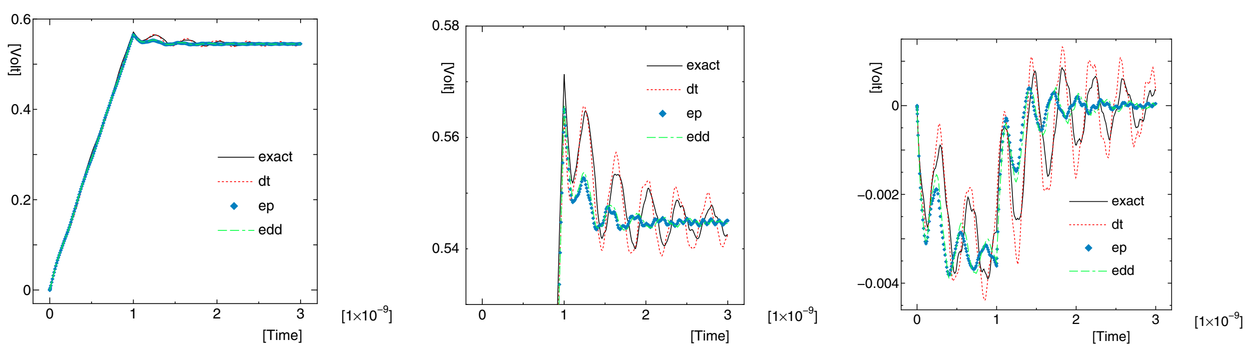


Fig. 1. System composed of PCB and package.



(a) (b) (c)

Fig. 5. Transient waveforms in using the sparse reluctance matrices generated with the threshold 10^9. (a)Response at a port of the package. (b)Part of 5(a). (c)Response at another port.



(a) (b) (c)

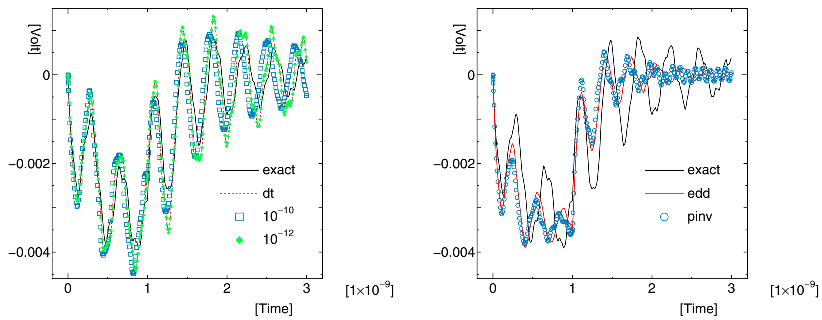
Fig. 6. Transient waveforms in using the sparse reluctance matrices generated with the threshold 10^8. (a)Response at a port of the package. (b)Part of 6(a). (c)Response at another port.

of the inverse matrix. As suggested in [1], [2], we can see that the inductance matrix becomes easier to be truncated than the original one of Fig. 2(a). In Figs. 4(a) and 4(b), '10^{-10}' and '10^{-12}' show that the inverse matrix is truncated by using these threshold values. When the threshold is 10^{-10}, the diagonal element is only selected in the 259-th row as shown in Fig. 4(b), that is, the inductance matrix has extremely sparse structure.

In many experiments, we found that a small threshold used in truncation of reluctance matrix tends to prevent from truncating the inverse matrix with guaranteed stability. Namely, the closer the inversion of the truncated reluctance matrix is to the original inductance matrix. the more difficult it is to truncate the inverse matrix stably. Then, we have to select a proper threshold, repeating the truncations.

B. SPICE Transient Analysis

Using the Berkeley SPICE (`ngspice-17`), we carried out the transient analysis of the system shown in Fig. 1, where the off-diagonal elements of the capacitance matrix having up to the third largest magnitude in each row was selected with the diagonal. Table 1 shows the CPU time comparison. In this table, 'threshold1' and 'threshold2' are threshold values used to truncate the reluctance and inductance matrices, respectively, and 'ratio1' and 'ratio2' show percentage of non-zero elements in the 500×500 and 493×493 reluctance matrices, respectively. In 'method' section, 'exact' shows



(a) (b)

Fig. 7. Transient waveforms obtained by the double inverse methods. (a)The result obtained by using the direct truncating. (b)The result obtained by using the Neumann series.

no truncated case, 'dt', 'ep', and 'edd' mean that direct truncating in Sect. III.A, enforcing positive definiteness in Sect. III.B, and enforcing diagonally dominance in Sect. III.C, and 'dt2' and 'pinv' imply that the sparse inductance matrices are obtained by the procedures of the sections IV.A and IV.B, respectively.

In Table 1, there are only small differences of CPU times among three methods in Sect. III which are about 400 and 20 times faster than no truncated case, depending on the threshold. When the threshold is 10^8, 'edd' is faster than 'dt' and 'ep'. This is because all the positive off-diagonal elements are dropped indifferent to the threshold. Hence, the 'ratio1' and 'ratio2' are lower than those of 'dt' and 'ep', which makes 'edd' faster than the others. Figs. 5(a)-6(c) show the transient waveforms at one of the 21 ports of the package or another port, where the sparse reluctance matrices are given with the threshold 10^9 in Figs. 5(a), 5(b), and 5(c), and 10^8 in Figs. 6(a), 6(b), and 6(c). By using a small threshold, it is expected to be given a more accurate result. In Fig. 6(c), an accurate result of 'dt' is certainly given, compared to the one in Fig. 5(c). However, the results of 'ep' and 'edd' do not approach the waveform of no truncated case, though a smaller threshold value is used. This would be due to an inherent error caused by adding magnitude of the off-diagonal elements to the corresponding diagonal ones.

Figures 7(a) and 7(b) show the transient waveforms obtained by the double inverse methods in Sect. IV. In Fig. 7(a), '10^{-10}' and 10^{-12}' show that the sparse inductance matrices are extracted with the threshold values 10^{-10} and 10^{-12}, respectively, after obtaining the reluctance matrix using the threshold 10^8. The needed CPU times are shown as 'dt2' of Table 1. The waveform of '10^{-10}' of Fig. 7(a) is very similar with the one of 'dt' which is the result obtained by the SPICE simulation which uses the reluctance matrix given by the direct truncating. Here, the needed CPU time is very small, and the simulation is approximately 2,000 times faster than no truncated case. From this fact, we can say that the double inverse method by using direct truncating is the most efficient of all the proposed methods.

In Fig. 7(b), the result of 'edd' is the same as in Fig. 6(c), and 'pinv' is the result obtained by the double inverse method in Sect. IV.B. To obtain the result, the first order Neumann series (11) is used, which means that the inductance matrix has the same sparse pattern as the reluctance matrix. While the result is almost identical to the result obtained by using the sparse reluctance matrix, the CPU time of the SPICE simulation is about 6 times faster than in using the reluctance matrix. Since the inversion is easy and the stability is guaranteed, the double inverse method using the pseudo inverse matrix of (11) by the Neumann series is profitable.

VI. Conclusions

In this paper, generating stable and sparse reluctance/inductance matrix has been presented in the case that the inductance extraction is done under insufficient discretization. Then, the reluctance matrix is not diagonally dominant nor M matrix, though positive definite. Here, some techniques how to generate the sparse reluctance matrix with guaranteed stability are proposed. Moreover, the sparse inductance matrix is obtained after truncating the off-diagonal elements of the inversion of the sparse reluctance matrix. These methods are also effectively used for obtaining the sparse inductance matrix when the extraction is done under sufficient discretization. In future, we will attempt to apply the proposed methods to the window-based reluctance/inductance extraction [1], [2], [5].

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